On the theory of single sampling inspection by attributes based on two quality levels.

By

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Institute of Mathematical Statistics, University of Copenhagen.
Technical Report No. 9 to the Office of Naval Research
prepared under Contract Nonr-N62558-3073 (NR 042-225).

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September 1965.

Contents.

Prepared with the partial support of the Office of Naval Research (Nonr-N62558-3073).

1. Introduction and summary.

Among the basic concepts in the theory of sampling inspection the producer's and the consumer's risks are the most widely used for characterizing system of sampling plans. It seems therefore strange that a comprehensive theory based on these concepts does not exist. The purpose of the present paper is to present such a theory for the case of single sampling by attributes. The theory naturally covers some well-known results, but old as well as new results are derived by a common method and compared within the same model. The requirements defining a system of sampling plans are usually of such a nature that no explicit solution exists for the sample size and the acceptance number. We shall therefore supplement the exact (implicit) solutions by asymptotic solutions which give a better insight into the basic properties of the systems.

Let there be given two quality levels, p_1 and p_2, $p_1 < p_2$. For a sampling plan, (n,c), n denoting the sample size and c the acceptance number, the operating characteristic is defined as $P(p) = B(c,n,p)$, where $B(c,n,p)$ denotes the cumulative binomial distribution. The producer's risk is then $Q(p_1) = 1-P(p_1)$, and the consumer's risk is $P(p_2)$. These risks give the probabilities of wrong decisions under the assumption that p_1 represents acceptable and p_2 rejectable quality.

We shall furthermore assume that the consequences of wrong decisions are commensurable and measurable and that the average "loss" from using a given sampling plan may be expressed as a linear combination of the two risks, $\gamma_1 Q(p_1) + \gamma_2 P(p_2)$ say. From a Bayesian point of view γ_i equals the product of the prior probability of p_i and the corresponding decision loss.

If the costs of sampling inspection are proportional to the sample size and we sample a lot of size N we may therefore write the average costs in the standard form

$$R(N,n,c) = n + (N-n)(\gamma_1 Q(p_1) + \gamma_2 P(p_2))$$

where $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. We shall use this cost function in the comparison of the various systems of sampling plans.

In section 2 it is shown how R may be interpreted as the average costs when (1) samples are drawn without replacement (2) from lots produced under binomial control but with a process average varying at random from p_1 to p_2, i.e. the prior distribution of lot quality is a double binomial, and (3) costs are linear in the number of defectives, the sample size and the lot size. Apart from a term of order e^{-N} the function R may thus be interpreted as the usual "risk" in decision theory. The model covers rectifying as well as non-rectifying inspection.

In sections 3 and 4 asymptotic expansions are derived for the producer's and the consumer's risk and for their ratio.

Section 5 gives a minimization theorem from which the relation between lot size and sample size may be found.

In the remaining part of the paper we discuss ten systems of sampling plans defined as follows:

(1). <u>Bayesian plans</u>, i.e. plans minimizing R.

<u>Restricted Bayesian plans</u>, i.e. plans minimizing R under some suitably chosen restriction on the operating characteristic, viz.

(2). Min R for $Q(p_1) = \alpha$ <u>or</u> $P(p_2) = \beta$.

(3). Min R for $P(p_0) = 1/2$ where $p_0 = \left(\log\frac{q_1}{q_2}\right) / \left(\log\frac{p_2 q_1}{p_1 q_2}\right)$.

(4). Min R for $Q(p_1) = \alpha/N$ <u>or</u> $P(p_2) = \beta/N$.

(5). Min R for $P(p_2)/Q(p_1) = \rho$.

<u>Plans defined by two risks</u>, viz.

(6). $Q(p_1) = \alpha/N$ <u>and</u> $P(p_2) = \beta/N$.

(7). $Q(p_1) = \alpha$ <u>and</u> $P(p_2) = \beta/N$ (or $P(p_2) = \beta$ <u>and</u> $Q(p_1) = \alpha/N$).

(8). $P(p_0) = 1/2$ <u>and</u> $Q(p_1) = \alpha/N$ (or $P(p_2) = \beta/N$).

(9). $Q(p_1) = \alpha$ <u>and</u> $P(p_2) = \beta$.

Finally we consider <u>percentage inspection</u> defined as

(10). $n = \mu N$ and $c = p_0 n$.

In all these definitions α, β, ρ, and μ represent suitably chosen positive constants which may be different from case to case.

For each system of sampling plans it is shown how the exact solution may be obtained and, since this solution is an implicit one, an explicit solution is given as an asymptotic expansion for $N \to \infty$.

The advantages of the asymptotic solution are that (1) it clearly shows how the sampling plan, the two risks, and the costs depend on the parameters, (2) it gives good approximations to the exact solution even for quite small values of c (normally sufficiently accurate for $c \geq 2$), (3) it may be used for developing interpolation and extrapolation formulas in connection with "master tables" of the exact solution, and (4) it shows the sensitivity of the solution with respect to changes of the parameters. We shall here mainly discuss the first of these points. With respect to the other three and a detailed discussion of tables the reader is referred to [9], [11], and [12].

The systems defined by (1) - (8) fall into two classes depending on whether both risks are $\underline{O}(N^{-1})$ or one of the risks is constant and the other is $\underline{O}(N^{-1})$.

The first class contains systems (1), (3), (4), (5), (6), and (8). It is proved that asymptotically the relation between acceptance number and sample size has the form $c = p_0 n + a_2 + a_4 n^{-1} + \underline{O}(n^{-2})$, that $\ln N = \varphi_0 n + \frac{1}{2}\ln n + \kappa_1 + \kappa_2 n^{-1} + \underline{O}(n^{-2})$, which by inversion determines n as a function of N, and that $R = n + \delta_1 + \delta_2 n^{-1} + \underline{O}(n^{-2})$, the constants p_0 and φ_0 being the same in all cases (depending on (p_1, p_2) only), whereas the remaining constants are found as functions of the parameters in the model and the restriction. This means that $n = \underline{O}(\ln N)$ and that the average decision loss, $(N-n)(\gamma_1 Q(p_1) + \gamma_2 P(p_2))$, tends to a constant (because $Q(p_1)$ and $P(p_2)$ are $\underline{O}(N^{-1})$).

The second class consists of systems (2) and (7). It is proved that asymptotically the relation between acceptance number and sample size has the form $c = p_j n + \sum_{i=1}^{4} a_i n^{1-i/2} + \underline{O}(n^{-3/2})$, p_j representing the quality level having a constant risk, that $\ln N = \varphi_j n + \frac{1}{2}\ln n + \sum_{i=1}^{4} \kappa_i n^{1-i/2} + \underline{O}(n^{-3/2})$, which determines n as a function of N, and that $R = \delta N + (1-\delta)n + \delta_1 + \underline{O}(n^{-1/2})$, the constants φ_j and δ being the same in the two cases. Because of the constant risk all relations are considerably more complicated than for the first class and R becomes $\underline{O}(N)$ instead of $\underline{O}(\ln N)$. For large lots it must therefore be seriously considered whether it is reasonable to use a system with a fixed consumer's or producer's risk and correspondingly high costs as compared with a system having decreasing risks.

The system with both risks fixed and the system with percentage inspection both lead to $R = \underline{O}(N)$ and asymptotically they have the same costs for $\mu = \gamma_1\alpha + \gamma_2\beta$. The system with fixed risks uses a fixed sample size so that the decision loss becomes of order N whereas percentage inspection has $n = \underline{O}(N)$ and a decision loss of order e^{-N}.

For systems (2) and (3) we have the important result that asymptotically n depends on (p_1, p_2) and $N\lambda$ only where λ is a function of γ_1, γ_2, and the parameter in the restriction. It therefore suffices to tabulate n as a function of N for $\lambda = 1$, say, and use this table for $\lambda \neq 1$ with $N^* = N\lambda$ as argument.

Writing B(c,np) for the cumulative Poisson distribution corresponding to B(c,n,p) we have under Poisson conditions exactly and under binomial conditions for small

(p_1, p_2) approximately that

$$Rp_1 = m + (M-m)(\gamma_1(1-B(c,m)) + \gamma_2 B(c,rm))$$

where $m = np_1$, $M = Np_1$, and $r = p_2/p_1$. Results analogous to those stated above are therefore valid in terms of M, m, and c for given (r, γ_1, γ_2), i.e. we save one parameter. It follows that under binomial conditions we have for systems (1), (2), (3), and (5) the following "proportionality law": The sampling plan corresponding to $(N, \lambda p_1, \lambda p_2)$ is approximately equal to $(n/\lambda, c)$ where (n,c) is the plan corresponding to $(N\lambda, p_1, p_2)$.

For systems (6) - (9) we have similarly: The sampling plan corresponding to $(N, \lambda p_1, \lambda p_2)$ is approximately equal to $(n/\lambda, c)$ where (n,c) is the plan corresponding to (N, p_1, p_2).

These theorems greatly reduce the tables necessary for applications of the systems.

It is shown that the restricted Bayesian plans with both risks decreasing and the corresponding plans based on two risks all have an economic efficiency tending to 1 for $N \to \infty$ as compared to the Bayesian plans. (The efficiency of plans having at least one risk fixed tends to zero). This result means that wrong values of the weights of the prior distribution and wrong values of the cost parameters have a secondary influence on the efficiency which tends to 1 if only (p_1, p_2) are correct. If also wrong values of (p_1, p_2), (p_1^*, p_2^*) say, are used for finding the plans, then the efficiency tends to e, $0 < e < 1$, if and only if $p_1 < p_1^* < p_2^* < p_2$, otherwise the efficiency tends to 0.

The present model leads to a constant ratio of the producer's and the consumer's risk for the (Bayesian) sampling plans. This provides a motivation for the rule of thumb suggested by Lehmann [14] for obtaining a reasonable balance between the probabilities of errors of the first and second kind in testing the hypothesis $p = p_1$ against the alternative $p = p_2$.

2. The model.

Let N and n denote lot size and sample size and let X and x denote number of defectives in the lot and the sample, respectively. The acceptance number is denoted by c.

Consider the following linear cost function

$$h(X,x,N,n,c) = \begin{cases} nS_1 + xS_2 + (N-n)A_1 + (X-x)A_2 & \text{for } x \leq c \\ nS_1 + xS_2 + (N-n)R_1 + (X-x)R_2 & \text{for } x > c \end{cases} \qquad (1)$$

and let the (prior) distribution of lot quality be $f_N(X)$. The average costs then become

$$K(N,n,c) = \sum_X \sum_x h(X,x)\, p(X,x)$$

where
$$p(X,x) = f_N(X)\binom{n}{x}\binom{N-n}{X-x}\Big/\binom{N}{X} = g_{N,n}(x)\, p(X|x),$$

$g_{N,n}(x)$ giving the (marginal) probability of getting x defectives in the sample, i.e.

$$g_{N,n}(x) = \binom{n}{x} \sum_{y=0}^{N-n} f_N(x+y)\binom{N-n}{y}\Big/\binom{N}{x+y}. \qquad (2)$$

This compound hypergeometric distribution has been discussed in [8] where the following results were proved:

$$E\left\{\frac{x^{(r)}}{n^{(r)}}\ \frac{(X-x)^{(s)}}{(N-n)^{(s)}}\right\} = E\left\{\frac{X^{(r+s)}}{N^{(r+s)}}\right\} \qquad (3)$$

and
$$E\left\{\frac{(X-x)^{(r)}}{(N-n)^{(r)}}\ \Big|\ x\right\} = \frac{(x+r)^{(r)}}{(n+r)^{(r)}}\ \frac{g_{N,n+r}(x+r)}{g_{N,n}(x)}\ . \qquad (4)$$

Writing $G_{N,n}(c) = \sum_{x=0}^{c} g_{N,n}(x)$ we find

$$K(N,n,c) = nS_1 + E(x)S_2 + (N-n)\left[A_1G_{N,n}(c) + R_1(1-G_{N,n}(c))\right]$$
$$+ A_2 \sum_{x=0}^{c} g_{N,n}(x)E(X-x|x) + R_2 \sum_{x=c+1}^{n} g_{N,n}(x)E(X-x|x). \qquad (5)$$

The last two terms of (5) may be simplified by using (4) for r = 1 which gives

$$g_{N,n}(x)E(X-x|x) = (N-n)\,\frac{x+1}{n+1}\, g_{N,n+1}(x+1)\ .$$

The average costs are obviously a rather complicated function of n and c and in general also of N. It is easy to see, however, that the necessary and sufficient condition for K to be a linear function of N for all values of (n,c) and all values of the cost parameters is that $g_{N,n}(x)$ is independent of N.

From g's independence of N follows that $E(x^{(r)}/n^{(r)})$ is independent of N so that (3) gives

$$E(X^{(r)}) = N^{(r)}\alpha_r \qquad \text{for } N \geq r, \qquad (6)$$

say, where α_r does not depend on N.

To find the class of distributions $f_N(X)$ satisfying (6) we introduce the limiting cumulative distribution of X/N defined by

$$W(p) = \lim_{N \to \infty} \sum_{x=0}^{[Np]} f_N(X) \ . \tag{7}$$

Proceeding as in [8] we find that

$$\alpha_r = \int_0^1 p^r dW(p)$$

and

$$f_N(X) = \binom{N}{X} \int_0^1 p^X q^{N-X} \, dW(p) \ . \tag{8}$$

It follows that $g_{N,n}(x) = f_n(x)$, i.e. the prior distribution is "reproduced" by hypergeometric sampling. (A discussion of reproducible distributions has been given in [8]).

We shall therefore get particularly simple results by <u>limiting the prior distributions to the class of distributions given by (8), called mixed binomial distributions</u>.

Writing $b(x,n,p) = \binom{n}{x} p^x q^{n-x}$ and

$$P(p) = B(c,n,p) = \sum_{x=0}^{c} b(x,n,p) \tag{9}$$

we get

$$\frac{x+1}{n+1} b(x+1,n+1,p) = p \, b(x,n,p)$$

and from (5)

$$K(N,n,c) = \int_0^1 K(N,n,c,p) dW(p) \tag{10}$$

where

$$K(N,n,c,p) = n(S_1+S_2p) + (N-n)\left[(A_1+A_2p)P(p) + (R_1+R_2p)Q(p)\right] . \tag{11}$$

The assumption of a mixed binomial prior distribution means that each lot is produced under binomial control and that the process average varies at random from lot to lot according to the cumulative distribution function $W(p)$. Correspondingly the average costs (11) represent an average over all lots with a given process average, i.e. a conditional average, and (10) gives the over-all average.

Besides giving the exact average costs for a mixed binomial prior distribution (10) and (11) may be interpreted as giving an approximation to the average costs for large N for any prior distribution satisfying (7). This follows from the facts that the hypergeometric distribution tends to the binomial for $N \to \infty$, $n \to \infty$, $n/N \to 0$ and $X = Np$, p fixed, and that $x = np + O(\sqrt{n})$, so that (1) becomes

$$h \sim \begin{cases} n(S_1 + S_2p) + (N - n)(A_1 + A_2p) \\ n(S_1 + S_2p) + (N - n)(R_1 + R_2p) \end{cases}$$

disregarding terms of order $\sqrt{n}$. The limit theorems derived in the following on the basis of (10) and (11) are therefore valid in general.

The model may easily be generalized from a linear to a polynomial cost function. Consider for example the term $(X - x)A_2$ which gives the contribution

$$(N - n)A_2 \int_0^1 p\, P(p)\, dW(p)$$

to the average costs. Introducing instead

$$\sum_{\nu=1}^{m} A_{2\nu} (X - x)^{(\nu)}$$

we find by using (4) the following average costs

$$\sum_{\nu=1}^{m} A_{2\nu}(N - n)^{(\nu)} \sum_{x=o}^{c} \frac{(x+\nu)^{(\nu)}}{(n+\nu)^{(\nu)}} g_{N,n+\nu} (x+\nu).$$

The condition for the average costs to be a polynomial in N is, as above, that g does not depend on N. For a mixed binomial prior distribution we get

$$\sum_{\nu=1}^{m} A_{2\nu}(N - n)^{(\nu)} \int_0^1 p^{\nu}P(p)\, dW(p).$$

Treating all six terms of (1) analogously we find the generalized average costs are given by (10) if we replace (11) by

$$K(N,n,c,p) = \sum_{\nu=1}^{m} n^{(\nu)}(S_{1\nu} + S_{2\nu}p^{\nu}) + \sum_{\nu=1}^{m} (N-n)^{(\nu)}\left[(A_{1\nu}+A_{2\nu}p^{\nu})P(p) + (R_{1\nu}+R_{2\nu}p^{\nu})Q(p)\right]. \quad (12)$$

Another generalization which is easily carried out consists of replacing (8) by

$$f_N(X) = \binom{N}{X} \int_0^1 p^X q^{N-X}\, dW_N(p) \quad (13)$$

where the comulative distribution function $W_N(p)$ depends on N. This will only result in a corresponding change of (10).

In the following we shall, however, mainly discuss the cost function defined by (10) and (11).

To simplify the notation we introduce the three cost functions

$$k_a(p) = A_1 + A_2 p, \quad k_r(p) = R_1 + R_2 p, \quad k_s(p) = S_1 + S_2 p, \tag{14}$$

and the corresponding averages k_a, k_r, and k_s, defined by

$$k = \int_0^1 k(p)dW(p). \tag{15}$$

If the equation $k_a(p) = k_r(p)$ has a solution $p = p_r$ in the interval (0,1) we define

$$k_m = \int_0^{p_r} k_a(p)dW(p) + \int_{p_r}^1 k_r(p)dW(p) \tag{16}$$

which represents the average costs per item when all lots from processes with $p \leq p_r$ are accepted and all other lots are rejected. The fraction defective p_r is called the (economic) break-even quality.

Defining the standardized form of (10) as

$$R(N,n,c) = (K(N,n,c) - Nk_m)/(k_s - k_m) \tag{17}$$

we find

$$R = n + (N-n)\frac{A_2 - R_2}{k_s - k_m}\left\{\int_0^{p_r}(p_r - p)Q(p)dW(p) + \int_{p_r}^1 (p - p_r)P(p)dW(p)\right\} \tag{18}$$

Using $k_s - k_m$ as "economic unit" the two terms of (18) represent the costs of sampling inspection and the average decision losses, respectively.

In the remainder of the paper it will be assumed that <u>the prior distribution is a double binomial distribution</u>, or as a limiting case, a single binomial. The double binomial distribution is a weighted average of two binomials with parameters p_1 and p_2, $p_1 < p_2$, and weights w_1 and w_2, $w_1 + w_2 = 1$, i.e. the process average has a two-point distribution.

<u>The standardized average costs may then be written as</u>

$$R(N,n,c) = n + (N-n)(\gamma_1 Q(p_1) + \gamma_2 P(p_2)) \tag{19}$$

where

$$\gamma_1 = w_1(k_r(p_1) - k_a(p_1))/(k_s - k_m) \text{ and } \gamma_2 = w_2(k_a(p_2) - k_r(p_2))/(k_s - k_m). \tag{20}$$

It will be noted that <u>the average decision loss per item is a linear combination of the producer's risk</u>, $Q(p_1)$, <u>and the consumer's risk</u>, $P(p_2)$.

The standardized costs of accepting or rejecting all lots without inspection are $R_a = N\gamma_2$ and $R_r = N\gamma_1$, respectively.

3. An asymptotic expansion for the binomial distribution.

There exist several asymptotic expansions for $1-B(c,n,p)$, $c = [np_0]$ and $p < p_0$, for $n \to \infty$, see for example Blackwell and Hodges [1] and Brockwell [2]. We need, however, an expansion under the assumption that $c/n = p_0 + \varepsilon$, $\varepsilon \to 0$ for $n \to \infty$, and shall use the same method as Brockwell to prove

Theorem 1. Let

$$c/n = p_0 + \sum_{i=1}^{4} a_i n^{-i/2} + \underline{O}(n^{-5/2}). \tag{21}$$

For $p > p_0$ we have

$$B(c,n,p) = \frac{q_0 p}{|p-p_0|\sqrt{2\pi n p_0 q_0}} \exp\left\{-n(\varphi(p_0,p) + \sum_{i=1}^{4} b_i(p_0,p)n^{-i/2} + \underline{O}(n^{-5/2}))\right\} \tag{22}$$

where

$$\varphi(p_0,p) = p_0 \ln\frac{p_0}{p} + q_0 \ln\frac{q_0}{q},$$

$$b_1 = a_1 \ln\frac{p_0 q}{q_0 p}, \qquad b_2 = a_2 \ln\frac{p_0 q}{q_0 p} + \frac{a_1^2}{2p_0 q_0},$$

$$b_3 = a_3 \ln\frac{p_0 q}{q_0 p} - \frac{a_1}{p-p_0} + \frac{a_1(1+2a_2)}{2p_0 q_0} - \frac{a_1^3(q_0-p_0)}{6(p_0 q_0)^2},$$

$$b_4 = a_4 \ln\frac{p_0 q}{q_0 p} - \frac{a_2}{p-p_0} - \frac{a_1^2 - 2p_0 q}{2(p-p_0)^2} + \frac{a_2 + a_2^2 + 2a_1 a_3}{2p_0 q_0}$$
$$+ \frac{1-p_0 q_0}{12 p_0 q_0} - \frac{(q_0-p_0)a_1^2(1+2a_2)}{4(p_0 q_0)^2} + \frac{(p_0^3+q_0^3)a_1^4}{12(p_0 q_0)^3}.$$

For $p < p_0$ the same expression is valid for $1-B(c,n,p)$.

Proof. Writing $c = nh$ and using Stirling's formula we get

$$\ln\binom{n}{c} = -\frac{1}{2}\ln(2\pi nh(1-h)) - n(h\ln h + (1-h)\ln(1-h)) - \frac{1-h(1-h)}{12nh(1-h)} + \underline{O}(n^{-3})$$

so that

$$\ln b(c,n,p) = -\frac{1}{2}\ln(2\pi nh(1-h)) - n\varphi(h,p) - \frac{1-h(1-h)}{12nh(1-h)} + \underline{O}(n^{-3})$$

where

$$\varphi(h,p) = h\ln\frac{h}{p} + (1-h)\ln\frac{1-h}{q}. \tag{23}$$

Expanding $\varphi(h,p)$ in a Taylor series around p_0 and inserting the given expression for $h-p_0$ we find

$$\varphi(h,p) = \varphi_0 + a_1\varphi_1 n^{-1/2} + (a_2\varphi_1 + \frac{1}{2}a_1^2\varphi_2)n^{-1} + (a_3\varphi_1 + a_1 a_2\varphi_2 + \frac{1}{6}a_1^3\varphi_3)n^{-3/2}$$
$$+ (a_4\varphi_1 + (\frac{1}{2}a_2^2 + a_1 a_3)\varphi_2 + \frac{1}{2}a_1^2 a_2\varphi_3 + \frac{1}{24}a_1^4\varphi_4)n^{-2} + \underline{O}(n^{-5/2})$$

where $\varphi_i = (\partial^i\varphi/\partial h^i)_{h=p_0}$. Similarly we have

$$\ln(h(1-h)) = \ln(p_0q_0) + \frac{(q_0-p_0)a_1}{p_0q_0\sqrt{n}} + \left(\frac{(q_0-p_0)a_2}{p_0q_0} - \frac{(p_0^2+q_0^2)a_1^2}{2(p_0q_0)^2}\right)\frac{1}{n} + \underline{O}(n^{-3/2})$$

and

$$\frac{1-h(1-h)}{12nh(1-h)} = \frac{1-p_0q_0}{12np_0q_0} + \underline{O}(n^{-3/2}) .$$

Combining these expressions we get an expansion for ln b(c,n,p) with remainder term of order $n^{-3/2}$.

Expressing the binomial by the incomplete Betafunction we have

$$B(c,n,p) = (n-c)\binom{n}{c}\int_0^q (1-x)^c x^{n-c-1}dx.$$

Changing the variable the integral becomes

$$\int_{1/q}^{\infty} (y-1)^c y^{-n-1}dy = \int_{1/q}^{\infty}\left(\frac{(y-1)^h}{y}\right)^n \frac{dy}{y} = \int_{z_0}^{\infty} e^{-nz}f(z)dz$$

where $z = -h\ln(y-1) + \ln y$ and $f(z) = y^{-1}dy/dz$ Integrating by parts leads to

$$\int_{z_0}^{\infty} e^{-nz}f(z)dz = \frac{1}{n}e^{-nz_0}\left(f(z_0) + \frac{1}{n}f'(z_0) + \underline{O}(n^{-2})\right).$$

As z_0 is found from z for $y = 1/q$ we have $z_0 = -h\ln p - (1-h)\ln q$ and $e^{-nz_0}/n = p^c q^{n-c}/n$. From

$$f(z) = (y-1)/(y(1-h)-1)$$

we get

$$f'(z) = -hy(y-1)/(y(1-h)-1)^3$$

so that

$$f(z_0) + \frac{1}{n}f'(z_0) = \frac{p}{p-h}\left(1-\frac{hq}{n(p-h)^2}\right).$$

Combining the results obtained we have

$$B(c,n,p) = b(c,n,p)\frac{(1-h)p}{p-h}\left(1-\frac{hq}{n(p-h)^2} + \underline{O}(n^{-2})\right) . \tag{24}$$

Expanding the logarithm of the factor to b(c,n,p) around p_0 we find

$$\ln\frac{B(c,n,p)}{b(c,n,p)} = \ln\frac{q_0p}{p-p_0} + \frac{a_1}{\sqrt{n}}\left(\frac{1}{p-p_0} - \frac{1}{q_0}\right)$$
$$+ \frac{1}{n}\left(\frac{a_2}{p-p_0} + \frac{a_1^2-2p_0q}{2(p-p_0)^2} - \frac{a_2}{q_0} - \frac{a_1^2}{2q_0^2}\right) + \underline{O}(n^{-3/2})$$

which result together with the expansion for $\ln b(c,n,p)$ after some reductions lead to the theorem.

A similar procedure leads to the result for $1-B(c,n,p)$.

It is obvious that the expansion may be continued by using the same method and that the following terms are of the same type as those given in theorem 1.

4. The ratio of the consumer's to the producer's risk.

From theorem 1 it follows that both the consumer's risk, $P(p_2) = B(c,n,p_2)$, and the producer's risk, $Q(p_1) = 1-B(c,n,p_1)$, tend exponentially to zero for $n \to \infty$ and $p_1 < p_0 < p_2$ since $\varphi(p_0,p_1) > 0$ and $\varphi(p_0,p_2) > 0$.

To discuss the ratio we introduce

$$\delta(p) = \varphi(p,p_1)-\varphi(p,p_2) = p \ln \frac{p_2 q_1}{p_1 q_2} - \ln \frac{q_1}{q_2}, \tag{25}$$

$\delta_0 = \delta(p_0)$ and $\delta' = \ln(p_2q_1/p_1q_2)$. It follows from theorem 1 that

$$\ln \frac{P(p_2)}{Q(p_1)} = \ln \frac{p_2(p_0-p_1)}{p_1(p_2-p_0)} + n\delta_0 + \sqrt{n}a_1\delta' + a_2\delta' + \frac{1}{\sqrt{n}}\left(a_3\delta' + \frac{a_1(p_2-p_1)}{(p_2-p_0)(p_0-p_1)}\right)$$

$$+ \frac{1}{n}\left(a_4\delta' + \frac{a_2(p_2-p_1)}{(p_2-p_0)(p_0-p_1)} + \frac{a_1^2-2p_0q_2}{2(p_2-p_0)^2} - \frac{a_1^2-2p_0q_1}{2(p_0-p_1)^2}\right) + O(n^{-3/2}). \tag{26}$$

For $\delta_0 \neq 0$ we find that $P(p_2)/Q(p_1) = O(e^{n\delta_0})$, so that one of the risks tends exponentially faster to zero than the other.

For $\delta_0 = 0$ and $a_1 = 0$ we have that $P(p_2)/Q(p_1)$ tends to a constant.

By means of the above expression we shall prove

<u>Theorem 2.</u> Let $c = np_0 + a_2$ and $p_0 = \left(\ln \frac{q_1}{q_2}\right) / \left(\ln \frac{p_2 q_1}{p_1 q_2}\right)$. Then

$$\frac{P(p_2)}{Q(p_1)} = \frac{p_2(p_0-p_1)}{p_1(p_2-p_0)} \left(\frac{p_2q_1}{p_1q_2}\right)^{a_2} \left\{1 + \frac{1}{c-a_2}\left(a_2\left(\frac{p_0}{p_2-p_0}+\frac{p_0}{p_0-p_1}\right) - \frac{q_2p_0^2}{(p_2-p_0)^2} + \frac{q_1p_0^2}{(p_0-p_1)^2}\right)\right\} + O(n^{-3/2}). \tag{27}$$

For small p_1 and p_2 and $a_2 = -2/3$ we have approximately $P(p_0) = 1/2$ and $P(p_2) = Q(p_1)$.

<u>Proof.</u> The first part of the theorem follows from (26) for $\delta_0 = a_1 = a_3 = a_4 = 0$.

The second part is found by letting $p_1 \to 0$ for fixed $r = p_2/p_1$. Noting that $p_0/p_1 \to (r-1)/\ln r$ and introducing

$$\varepsilon_1 = \frac{r-1}{r-1-\ln r} \quad \text{and} \quad \varepsilon_2 = \frac{r-1}{r \ln r - r + 1}$$

we find

$$\frac{P(p_2)}{Q(p_1)} \longrightarrow r^{a_2+1}\ \frac{g_2}{g_1}\left\{1 + \frac{1}{c \cdot a_2}\,(g_1+g_2)(a_2+g_1-g_2)\right\} + \underline{O}(n^{-3/2})$$

which for $r < 20$ is approximately equal to

$$\frac{P(p_2)}{Q(p_1)} \approx r^{a_2+2/3}\left\{1 + \frac{1}{c-a_2}\,(g_1+g_2)(a_2+\tfrac{2}{3})\right\}. \qquad (28)$$

The last result rests on the (numerical) fact that $r^{1/3}\ g_2/g_1 \approx 1$ and $g_1 - g_2 \approx 2/3$ for $r < 20$ as will be seen from the following table.

<u>Table of g_1, g_2, and $r^{1/3}\ g_2/g_1$</u>

r	g_1	g_2	g_1-g_2	g_1+g_2	$r^{1/3}\ g_2/g_1$
1	∞	∞	0.67	∞	1.00
2	3.26	2.59	0.67	5.85	1.00
3	2.22	1.54	0.68	3.76	1.00
5	1.67	0.99	0.68	2.66	1.01
7	1.48	0.79	0.69	2.27	1.02
10	1.34	0.64	0.70	1.98	1.03
15	1.24	0.53	0.71	1.77	1.05
20	1.19	0.46	0.73	1.65	1.06

Since the equation $B(c,n,p_0) = 1/2$ has the solution $c = np_0 - (2 - p_0)/3 + \underline{O}(n^{-1})$, see section 9, the last part of the theorem is proved.

Theorem 2 implies that a system of sampling plans defined (partially) by the relation $c = np_0 + a_2$ will have a ratio $P(p_2)/Q(p_1)$ tending decreasingly or increasingly to a constant according as a_2 is larger or smaller than $-2/3$ respectively. If p_0 differs from the value defined in theorem 2 then $P(p_2)/Q(p_1)$ will tend exponentially to zero or infinity..

From (26) we also find

<u>Theorem 3.</u> Let $c/n = p_0 + \sum_{i=1}^{4} a_i n^{-i/2} + \underline{O}(n^{-5/2})$. Then $P(p_2)/Q(p_1)$ is constant (to the order of approximation here considered), i.e.

$$\frac{P(p_2)}{Q(p_1)} = \frac{p_2(p_0-p_1)}{p_1(p_2-p_0)}\left(\frac{p_2q_1}{p_1q_2}\right)^{a_2}\left(1 + \underline{O}(n^{-3/2})\right) \qquad (29)$$

if and only if $p_0 = (\ln\frac{q_1}{q_2})/\ln\left(\frac{p_2q_1}{p_1q_2}\right)$, $a_1 = a_3 = 0$, and

$$a_4 = -\frac{1}{8'}\left(\frac{a_2}{p_0-p_1} + \frac{a_2}{p_2-p_0} + \frac{p_0q_1}{(p_0-p_1)^2} - \frac{p_0q_2}{(p_2-p_0)^2}\right). \tag{30}$$

For small p_1 and p_2 we have approximately

$$c \approx np_0 + a_2 - (g_1+g_2)(a_2+2/3)/(np_0 \ln r)$$

and $P(p_2)/Q(p_1) \approx r^{a_2+2/3}$ where $r = p_2/p_1$.

Proof. The main result follows directly from (26) and the approximation is found by similar considerations as under theorem 2.

5. A minimization theorem.

For use in later sections we need

Theorem 4. The minimum of

$$R(n) = n + (N-n)\lambda n^{-1/2} f(n) \tag{31}$$

where $f(n) = \exp(-n \sum_{i=0}^{4} \alpha_i n^{-i/2})$, $\lambda > 0$ and $\alpha_0 > 0$, with respect to n for $N \to \infty$ is equal to

$$\min R = n + \frac{1}{\alpha_0}\left(1 - \frac{\alpha_1}{2\alpha_0\sqrt{n}} + \frac{\alpha_1^2 - 2\alpha_0}{4\alpha_0^2 n}\right) + O(n^{-3/2}), \tag{32}$$

n being determined as a function of N from the equation

$$\ln(N-n) = -\ln\lambda + \frac{1}{2}\ln n + n\sum_{i=0}^{4} \alpha_{i0}\, n^{-i/2} + O(n^{-3/2}) \tag{33}$$

where $\alpha_{00} = \alpha_0$, $\alpha_{10} = \alpha_1$, $\alpha_{20} = \alpha_2 - \ln\alpha_0$, $\alpha_{30} = \alpha_3 - \alpha_1/2\alpha_0$, and $\alpha_{40} = \alpha_4 - 1/2\alpha_0 + \alpha_1^2/8\alpha_0^2$.

Proof. From

$$R'(n) = 1 - \lambda n^{-1/2} f(n) + (N-n)(\lambda n^{-1/2} f'(n) - \tfrac{1}{2}\lambda n^{-3/2} f(n)) = 0$$

we get

$$(N-n)\lambda n^{-1/2} f(n) = \left(-\frac{f'(n)}{f(n)} + \frac{1}{2n}\right)^{-1}\left(1 - \lambda n^{-1/2} f(n)\right).$$

It follows that $n \to \infty$ for $N \to \infty$.

Since

$$-\frac{f'(n)}{f(n)} + \frac{1}{2n} = \alpha_0 + \frac{\alpha_1}{2\sqrt{n}} + \frac{1}{2n} + O(n^{-3/2})$$

and $1 - \lambda n^{-1/2} f(n) = 1 + \underline{O}(e^{-n})$ we find

$$(N - n)\lambda n^{-1/2} f(n) = \left(\alpha_0 + \frac{\alpha_1}{2\sqrt{n}} + \frac{1}{2n} + \underline{O}(n^{-3/2})\right)^{-1}$$

$$= \frac{1}{\alpha_0}\left(1 - \frac{\alpha_1}{2\alpha_0\sqrt{n}} + \frac{\alpha_1^2 - 2\alpha_0}{4\alpha_0^2 n}\right) + \underline{O}(n^{-3/2}) \qquad (34)$$

which immediately gives min R. Taking logarithms on both sides of the last equation we obtain the equation for determining n.

Inversion of this equation gives

$$\alpha_0 n = x - \beta_1 x^{1/2} - \tfrac{1}{2}\ln x + \beta_2 + \tfrac{1}{4}\beta_1 x^{-1/2}\ln x + \beta_3 x^{-1/2} + \tfrac{1}{4}x^{-1}\ln x + \beta_4 x^{-1} + o(x^{-1}), \quad (35)$$

where $x = \ln N$, $\beta_1 = \alpha_1\alpha_0^{-1/2}$, $\beta_2 = \frac{1}{2}\beta_1^2 - \alpha_2 + \frac{3}{2}\ln\alpha_0 + \ln\lambda$,

$\beta_3 = \beta_1 - \frac{1}{8}\beta_1^3 + \frac{1}{2}\beta_1\alpha_2 - \alpha_0^{1/2}\alpha_3 - \frac{3}{4}\beta_1 \ln\alpha_0 - \frac{1}{2}\beta_1 \ln\lambda$,

$\beta_4 = \frac{1}{2} - \frac{3}{8}\beta_1^2 + \frac{1}{2}\alpha_2 - \alpha_0\alpha_4 - \frac{3}{4}\ln\alpha_0 - \frac{1}{2}\ln\lambda$,

which determines the value of n in min R.

The importance of theorem 4 is due to the fact that the asymptotic form of R(N,n,c) after elimination of $c = np_0 + \sum_{i=1}^{4} a_i n^{1-i/2}$ is equal to (31). The result (32) then tells that the minimum standardized costs asymptotically consist of sampling inspection costs plus a term tending to a constant which represents the limiting average decision losses for the remainder of the lot.

An important corollary is found by noting from (33) that asymptotically n depends only on $\ln(N\lambda)$, i.e. on the product of the lot size and the parameter λ. If sample size has been tabulated as function of lot size for one value of λ, $\lambda = 1$ say, we may therefore use the same table to find the sample size corresponding to lot size N and any λ by using $N^* = N\lambda$ as argument.

5. Bayesian single sampling plans.

The Bayesian solution consists of determining the value of (n,c) minimizing R(N,n,c) and using this sampling plan if min R is less than the costs of accepting or rejecting all lots without inspection. A necessary condition for a sampling plan to exist is that $\gamma_1 > 0$ and $\gamma_2 > 0$, i.e. $p_1 < p_r < p_2$.

Values of (n,c) minimizing R must satisfy the two inequalities

$$\Delta_c R(N,n,c-1) \leq 0 < \Delta_c R(N,n,c), \qquad 0 \leq c \leq n,$$

and

$$\Delta_n R(N,n-1,c) \leq 0 < \Delta_n R(N,n,c), \qquad c \leq n \leq N,$$

Δ denoting the usual forward difference operator. Noting that $\Delta_c B(c,n,p) = b(c+1,n,p)$ and $\Delta_n B(c,n,p) = -pb(c,n,p)$ we find from (19)

$$\Delta_c R(N,n,c) = (N-n)(-\gamma_1 b(c+1,n,p_1) + \gamma_2 b(c+1,n,p_2))$$

and

$$\Delta_n R(N,n,c) = 1 - (\gamma_1 Q(p_1) + \gamma_2 P(p_2)) + (N-n-1)(\gamma_1 p_1 b(c,n,p_1) - \gamma_2 p_2 b(c,n,p_2)).$$

Solving the inequalities with respect to n and N we find that a Bayesian sampling plan must satisfy the two inequalities

$$\alpha + \beta c \leq n < \alpha + \beta(c+1) \tag{36}$$

and

$$F(n-1,c) \leq N < F(n,c) \tag{37}$$

where

$$\alpha = (\ln \frac{\gamma_2}{\gamma_1})/(\ln \frac{q_1}{q_2}), \tag{38}$$

$$\beta = (\ln \frac{p_2 q_1}{q_2 p_1}) / (\ln \frac{q_1}{q_2}), \tag{39}$$

and

$$F(n,c) = n + 1 + \frac{1-\gamma_1+\gamma_1 B(c,n,p_1) - \gamma_2 B(c,n,p_2)}{-\gamma_1 p_1 b(c,n,p_1) + \gamma_2 p_2 b(c,n,p_2)}.$$

For two plans (n_1,c_1) and (n_2,c_2), $c_1 < c_2$ say, satisfying (36) and having overlapping N-intervals according to (37) the cost functions must be compared. Solving the equation $R(N,n_1,c_1) = R(N,n_2,c_2)$ for N we get

$$N_{12} = \frac{(n_2-n_1)(1-\gamma_1) + n_2\gamma(n_2,c_2) - n_1\gamma(n_1,c_1)}{\gamma(n_2,c_2) - \gamma(n_1,c_1)}$$

where

$$\gamma(n,c) = \gamma_1 B(c,n,p_1) - \gamma_2 B(c,n,p_2).$$

Since R for given (n,c) is an increasing linear function of N we have that $R(N,n_1,c_1) \lesseqgtr R(N,n_2,c_2)$ according as $N \lesseqgtr N_{12}$.

It will be noted that the simplicity of the solution depends essentially on R(N,n,c) being a linear function of N for given (n,c).

The above solution has previously been given in [8] and a rather complete tabulation of Bayesian sampling plans has been provided in [11].

The exact solution given above does not disclose the structure of the relationships between N,n, and c. We shall therefore derive an asymptotic solution.

Setting $c/n = p_0 + \varepsilon$, $\varepsilon \to 0$ for $n \to \infty$, we shall first find p_0 and ε and afterwards determine the relation between N and n by means of theorem 4.

Using (24) and the expansion of b(c,n,p) we get for $p_0 < p$

$$B(c,n,p) = \frac{q_0 p}{|p-p_0|\sqrt{2\pi n p_0 q_0}} \exp\{-n\varphi(p_0+\varepsilon,p) + \underline{O}(\varepsilon) + \underline{O}(n^{-1})\}$$

and the same expression for $1 - B(c,n,p)$ for $p_0 > p$. For $p_1 < p_0 < p_2$ we therefore have the following asymptotic expansion for the cost function

$$R = n + (N-n)n^{-1/2}\sum_{i=1}^{2}\lambda_i \exp[-n\varphi(p_0,p_i) - n\varepsilon\varphi_i' + \underline{O}(\varepsilon) + \underline{O}(n^{-1})] \tag{40}$$

where

$$\lambda_i = q_0 p_i \gamma_i \;/\; |p_0-p_i|\sqrt{2\pi p_0 q_0} \tag{41}$$

and

$$\varphi_i' = \varphi'(p_0,p_i) = \ln(p_0 q_i/q_0 p_i). \tag{42}$$

By means of this expression and theorem 4 we shall prove

<u>Theorem 5</u>. For the Bayesian single sampling plans we have

$$c = np_0 + a_2 + a_4/n + \underline{O}(n^{-2}) \tag{43}$$

where

$$p_0 = \left(\ln\frac{q_1}{q_2}\right) \Big/ \left(\ln\frac{p_2 q_1}{p_1 q_2}\right), \tag{44}$$

$$a_2 = \frac{1}{\delta'}\ln\frac{\lambda_1\varphi_1'}{\lambda_2(-\varphi_2')} \quad\text{and}\quad a_4 = -\frac{1}{\delta'}\left(\frac{a_2}{p_0-p_1} + \frac{a_2}{p_2-p_0} + \frac{p_0 q_1}{(p_0-p_1)^2} - \frac{p_0 q_2}{(p_2-p_0)^2}\right). \tag{45}$$

Further

$$\min R = n + \frac{1}{\varphi_0}\left(1 - \frac{1}{2\varphi_0 n}\right) + \underline{O}(n^{-2}), \tag{46}$$

$$\ln(N-n) = \varphi_0 n + \tfrac{1}{2}\ln n + a_2\varphi_1' - \ln(\varphi_0\lambda_1\delta'/1-\varphi_2') + (a_4\varphi_1' + \beta_1 - 1/2\varphi_0)\frac{1}{n} + \underline{O}(n^{-2}) \tag{47}$$

$$Q(p_1) = \frac{-\varphi_2'}{\varphi_0\gamma_1\delta'}\left(1 - \frac{1}{2\varphi_0 n} + \underline{O}(n^{-2})\right)\frac{1}{N-n} \tag{48}$$

and

$$P(p_2) = \frac{\varphi_1'}{\varphi_0\gamma_2\delta'}\left(1 - \frac{1}{2\varphi_0 n} + \underline{O}(n^{-2})\right)\frac{1}{N-n} \tag{49}$$

where

$$\varphi_0 = p_0 \ln \frac{p_0}{p_i} + q_0 \ln \frac{q_0}{q_i}\,, \qquad i = 1 \text{ or } 2\,, \tag{50}$$

and

$$\beta_i = \frac{a_2^2+a_2}{2p_0q_0} + \frac{1-p_0q_0}{12p_0q_0} - \frac{a_2}{p_i-p_0} + \frac{p_0q_i}{(p_i-p_0)^2}\,. \tag{51}$$

Proof. For $p_1 < p_0 < p_2$ it follows from (40) that $R/N \longrightarrow 0$ for $N \longrightarrow \infty$, $n \longrightarrow \infty$, and $n/N \longrightarrow 0$. For $p_0 < p_1$ we have that $Q(p_1) \longrightarrow 1$ and $P(p_2) \longrightarrow 0$ so that $R/N \longrightarrow \gamma_1 > 0$. For $p_0 = p_1$ we may use the normal approximation which gives that $Q(p_1) \longrightarrow \alpha$, $0 < \alpha < 1$, so that $R/N \longrightarrow \alpha\gamma_1 > 0$. The optimum value of p_0 satisfies consequently the inequality $p_1 < p_0 < p_2$.

It is easy to see (indirectly) that the optimum value of p_0 must satisfy the equation $\varphi(p_0,p_1) = \varphi(p_0,p_2)$ which leads to (44) because any other choice of p_0 will make one of the exponential terms in (40) larger than $\exp(-n\varphi_0)$.

To find a first approximation to ε we put $dR/d\varepsilon = 0$ which gives

$$\sum_{i=1}^{2} \lambda_i \varphi_i' \, e^{-n\varepsilon\varphi_i'} = 0.$$

Solving for $a_2 = n\varepsilon$ we get the first part of (45), i.e. $c = np_0 + a_2 + o(1)$ as could be expected from (36) since $p_0 = 1/\beta$.

To get one more term we write $c/n = p_0 + (a_2+\varepsilon)/n$, $\varepsilon \longrightarrow 0$, and repeat the procedure to find the new ε. Using the same method as in section 3 we may develop (40) into

$$R = n + (N-n)n^{-1/2} \sum_{i=1}^{2} \lambda_i \exp\{-n\varphi_0 - a_2\varphi_i' - (n\varepsilon\varphi_i' + \beta_i)/n + \underline{O}(\varepsilon/n) + \underline{O}(n^{-2})\}$$

where β_i has been defined in (51). From $dR/d\varepsilon = 0$ we find the equation $n\varepsilon\varphi_1' + \beta_1 = n\varepsilon\varphi_2' + \beta_2$. Solving for $a_4 = n\varepsilon$ we get the second part of (45).

The remaining part of the theorem is a direct consequence of theorem 4 for
$\alpha_0 = \varphi_0$, $\alpha_2 = a_2\varphi_1' - \ln(\lambda_1\varphi_1') = a_2\varphi_2' - \ln(-\lambda_2\varphi_2')$,
$\alpha_4 = a_4\varphi_1' + \beta_1 = a_4\varphi_2' + \beta_2$, and $\lambda = \delta/(-\varphi_1'\varphi_2')$.

It will be noted that asymptotically the ratio between the consumer's and the producer's risk is constant for the Bayesian plans, viz.

$$P(p_2)/Q(p_1) = \gamma_1\varphi_1'/\gamma_2(-\varphi_2'), \tag{52}$$

cf. Theorem 3.

From (36) one could hope that $n \cong \alpha + \beta(c + \frac{1}{2})$ and consequently that $a_2 \cong -p_0\alpha - 1/2$. In [11] it has been demonstrated numerically that this approximation is rather good. We shall now derive the limiting values of a_2 and a_4 for $p_1 \longrightarrow 0$ and $r = p_2/p_1$ fixed, and use the result to discuss the approximation above.

Proceeding as in section 4 we find

$$a_2 \longrightarrow \frac{\ln(\gamma_1/\gamma_2)}{\ln r} - 1 + \frac{1}{\ln r} \ln \frac{g_1 \ln(g_1/(g_1-1))}{g_2 \ln((g_2+1)/g_2)}$$

and

$$p_0 a_4 \longrightarrow -(g_1 + g_2)(a_2 + g_1 - g_2)/\ln r.$$

The last term in lim a_2 decreases from 0.50 to 0.47 as r increases from 1 to 20, i.e. the last term is practically equal to 1/2.

For small p_1 and p_2 we therefore have approximately

$$c = np_0 + \frac{\ln(\gamma_1/\gamma_2)}{\ln r} - \frac{1}{2} \cdot \frac{1}{np_0 \ln r}(g_1+g_2)(a_2 + 2/3) + \underline{O}(n^{-2}).$$

The fraction of the average decision loss which is "due to" the consumer's risk equals φ_1'/δ'. This fraction tends to $(\ln(r-1) - \ln \ln r)/\ln r$ for $p_1 \longrightarrow 0$ ($r = p_2/p_1$ fixed), and increases from 0.50 to 0.62 as r increases from 1 to 20.

7. Restricted Bayesian sampling plans.

One of the objections against the Bayesian solution is that it does not always lead to a sampling plan, particularly for small lots. In such cases a running check on the assumptions regarding the prior distribution is lacking and, if quality deteriorates, the delay before appropriate measures can be taken may be excessive.

There may also be cases where a producer, say, inspecting his own goods sets an upper limit for the probability of passing bad lots.

It is therefore useful to study restricted Bayesian solutions derived by minimizing average costs under a suitably chosen restriction. Such restrictions may be of an economical, technical, or statistical nature. We shall here, however, only consider restrictions on the operating characteristic, i.e. restrictions which are independent of the weights in the prior distribution and the cost functions.

The first restriction of that kind was introduced by Dodge and Romig [4] in their LTPD system of sampling plans. For the present model it consists in specifying $P(p_2) = \beta$, where β customarily is chosen as 0.10. Correspondingly, one may choose to specify $Q(p_1) = \alpha$ with $\alpha = 0.05$, say.

It follows from the properties of the Bayesian sampling plans that, at least for large lots, it will be uneconomical to specify a fixed risk for the consumer or the producer. Instead, the risk should be chosen as a decreasing function of lot size, for example as $P(p_2) = \beta_0$ for $N \leq N_0$ and $P(p_2) = \beta_0 N_0/N$ for $N > N_0$, or similarly as $Q(p_1) = \alpha_0$ for $N \leq N_0$ and $Q(p_1) = \alpha_0 N_0/N$ for $N > N_0$, which asymptotically corresponds to (48) and (49).

Another possibility is to specify $P(p_0) = 1/2$ for $p_1 < p_0 < p_2$ which will lead to decreasing risks both for the producer and the consumer. If p_0 is chosen as in (44) both risks will be $\underline{O}(1/N)$, otherwise they will tend to zero with different rates of convergence.

Finally one may specify the ratio between the two risks, i.e. $Q(p_1) = \rho P(p_2)$, which also results in decreasing risks with increasing lot size. The corresponding Bayesian result is given by (52).

It will be seen that it is possible (inspired by the asymptotic properties of the Bayesian sampling plans) to introduce restrictions on the operating characteristic which will change the Bayesian solution for small lots in a desired direction and at the same time preserve the properties of the Bayesian solution for large lots.

It should be noted that restricted Bayesian plans exist also if one of the parameters γ_1 and γ_2 is zero which means that the double binomial prior distribution degenerates to a single binomial. The other quality level is then introduced through the restriction with the purpose to avoid too heavy losses if the prior distribution changes. In sections 8 - 11 we shall discuss the properties of the various types of restricted Bayesian sampling plans.

8. Minimum average costs for fixed consumer's or producer's risk.

We shall limit the mathematical discussion to the case with a fixed consumer's risk since the two cases are completely analogous.

Inserting $P(p_2) = \beta$ into (19) we get

$$R(N,n,c) = n(1 - \gamma_2\beta) + (N - n)\gamma_1 Q(p_1) + N\gamma_2\beta$$

$$= (1 - \gamma_2\beta)R_1 + N\gamma_2\beta \qquad (53)$$

where

$$R_1 = n + (N - n)\gamma Q(p_1), \qquad \gamma = \gamma_1/(1 - \gamma_2\beta). \qquad (54)$$

The problem is to minimize R or equivalently R_1 under the restriction

$$P(p_2) = B(c,n,p_2) = \beta. \tag{55}$$

Since (55) defines a relation between n and c, $n = n_c$ say, (which is unique if n is determined as the smallest integer satisfying $B(c,n,p_2) \leq \beta$) we may write R_1 as a function of c

$$R_1(c) = n_c + (N - n_c)\gamma(1 - B_c)$$

where $B_c = B(c,n_c,p_1)$. The value of c minimizing R_1 is determined from the inequality $\Delta R_1(c-1) \leq 0 < \Delta R_1(c)$. As

$$\Delta R_1(c) = (1 - \gamma + \gamma B_c)\Delta n_c - \gamma(N - n_{c+1})\Delta B_c$$

we define the auxiliary function

$$N_c = n_{c+1} + (1/\gamma - 1 + B_c)\Delta n_c/\Delta B_c$$

so that (n_c,c) is the optimum plan for lot size N if $N_{c-1} \leq N < N_c$, $\Delta B_{c-1} > 0$ and $\Delta B_c > 0$.

These formulas are well suited for a systematic tabulation since n_c and N_c are easily found for $c = 0,1,2,\ldots$.

The above solution has previously been discussed and tabulated in [12]. A similar approach has been used in [9], and by Dodge and Romig [4]. It will be noted that the Dodge Romig LTPD plans are obtained for $\gamma_1 = 1$ and $\gamma_2 = 0$ with the modification that Dodge and Romig use the hypergeometric distribution instead of the binomial in defining the restriction.

The asymptotic solution is given in

Theorem 6. The relation between acceptance number and sample size is given by

$$c/n = p_2 + \sum_{i=1}^{4} a_i n^{-i/2} + O(n^{-5/2}) \tag{56}$$

where

$$a_1 = -u(p_2q_2)^{1/2}, \qquad a_2 = -\frac{1}{2} + \frac{1}{6}(q_2-p_2)(u^2-1),$$

$$a_3 = -\frac{1}{24}(1-6p_2q_2)(p_2q_2)^{-1/2}(u^3-3u) + \frac{1}{36}(q_2-p_2)^2(p_2q_2)^{-1/2}(2u^3-5u),$$

$$a_4 = \frac{q_2-p_2}{p_2q_2}\left\{\frac{1}{120}(1-12p_2q_2)(u^4-6u^2+3) - \frac{1}{24}(1-6p_2q_2)(u^4-5u^2+2) + \frac{1}{324}(q_2-p_2)^2(12u^4-53u^2+17)\right\},$$

and u denotes the (1-β)-fractile of the standardized normal distribution.

The relation between sample size and lot size is given by (33) or (35), and

$$\min R = (1-\gamma_2\beta)\left\{n + \frac{1}{\alpha_0}\left(1 - \frac{\alpha_1}{2\alpha_0\sqrt{n}} + \frac{\alpha_1^2 - 2\alpha_0}{4\alpha_0^2 n}\right)\right\} + \gamma_2\beta N + O(n^{-3/2}) \quad (57)$$

and

$$Q(p_1) = \frac{1-\gamma_2\beta}{\gamma_1\alpha_0}\left(1 - \frac{\alpha_1}{2\alpha_0\sqrt{n}} + \frac{\alpha_1^2 - 2\alpha_0}{4\alpha_0^2 n} + O(n^{-3/2})\right)\frac{1}{N-n}, \quad (58)$$

where $\alpha_0 = \varphi(p_2, p_1)$, $\alpha_i = b_i(p_2, p_1)$ for $i \geq 1$,

and $\lambda = (\gamma q_2 p_1)/((p_2 - p_1)\sqrt{2\pi p_2 q_2})$.

<u>Proof</u>. The asymptotic solution to (55) may be found from the Fisher-Corrish [6] expansion which leads to (56) since $\kappa_3/\sigma^2 = q - p$, $\kappa_4/\sigma^2 = 1 - 6pq$, and $\kappa_5/\sigma^2 = (q-p)(1-12pq)$ for the binomial. The u-polynomials have been tabulated as functions of β in [6].

The remaining part of the proof consists of a direct application of theorems 1 and 4.

Analogous results for the case with $\gamma_1 = 1$, $\gamma_2 = 0$ and Poisson probabilities have been given in [9] which also contains an evaluation of the accuracy of the asymptotic solution. In the present paper the expansion has, however, been carried two terms further than in [9] and [12].

Since the producer's risk tends (exponentially) to zero and the consumer's risk is fixed the average decision loss tends to a constant plus $\gamma_2\beta N$. This last term will for large N become dominating also as compared to the costs of sampling (which are of order ln N) and for this reason plans with fixed consumer's or producer's risk are uneconomical for large N.

An important result is that asymptotically <u>the sampling plan depends only on the product of lot size and cost parameter</u> so that the plan for lot size N and cost parameter γ equals the plan for lot size Nγ and cost parameter 1. Since this property holds with good approximation also for small N, tabulation of such plans may be limited to γ = 1 and γ = 5, say, for LTPD plans, and to γ = 1 and γ = 0.2 for AQL plans.

The present LTPD model generalizes the Dodge-Romig system of LTPD plans in two respects: (1) the single binomial prior distribution has been replaced by a double binomial, (2) the simple cost function $I = n + (N-n)Q(p_1)$, which pertains to rectifying inspection with the same costs for sampling inspection and sorting, has been replaced by R which allows a much broader interpretation including rectifying as well as non-rectifying inspection.

In case $w_2 = 0$, i.e. $\gamma_2 = 0$, the quality level p_2 is introduced through the restriction alone, which means that p_2 has to be determined from technical and economical considerations and not from the prior distribution.

A detailed discussion of both LTPD and AQL plans and a comparison with other systems has been given in [12].

9. Minimum average costs for $P(p_0) = 1/2$.

The restriction

$$P(p_0) = B(c,n,p_0) = 1/2, \quad p_1 < p_0 < p_2,$$

defines a relation between n and c, $n = n_c$ say. Proceeding as in section 8 we find that the plan (n_c, c) is optimum for $N_{c-1} \leq N < N_c$ where

$$N_c = n_{c+1} - (1-\gamma_1 Q(p_1) - \gamma_2 P(p_2))\Delta n_c / (\gamma_1 \Delta Q(p_1) + \gamma_2 \Delta P(p_2)),$$

$$Q(p_1) = 1 - B(c,n_c,p_1) \quad \text{and} \quad P(p_2) = B(c,n_c,p_2).$$

In principle such IQL plans may be defined for any value of p_0. If no (technical) reasons exist for choosing a specific value of p_0 it seems reasonable to determine p_0 such that R is minimized. From (26) it follows that one of the risks will be infinitely small as compared to the other for $n \to \infty$ depending on which of the two coefficients $\varphi(p_0,p_1)$ and $\varphi(p_0,p_2)$ is the larger. If $\varphi(p_0,p_2) \geq \varphi(p_0,p_1)$ we find that min $R \sim n + 1/\varphi(p_0,p_1)$,

$$Q(p_1) = \underline{O}(1/N), \quad \text{and} \quad P(p_2) = \underline{O}(N^{-\varphi(p_0,p_2)/\varphi(p_0,p_1)}).$$

Since $n \sim (\ln N)/\varphi(p_0,p_1)$ it follows that min R will be minimized with respect to p_0 for $\varphi(p_0,p_1)$ being as large as possible, i.e. for $\varphi(p_0,p_1) = \varphi(p_0,p_2)$, which leads to p_0 as defined by (44). We shall therefore only study IQL plans for this value of p_0.

<u>Theorem 7.</u> The relation between acceptance number and sample size is given by

$$c = np_0 - \frac{2-p_0}{3} - \frac{(q_0-p_0)(19+32p_0q_0)}{3240np_0q_0} + \underline{O}(n^{-2}). \tag{59}$$

The relation between sample size and lot size is given by

$$\ln(N-n) = n\varphi_0 + \frac{1}{2}\ln n - \ln(\lambda\varphi_0) + (b_4 - \frac{1}{2\varphi_0})\frac{1}{n} + \underline{O}(n^{-2}), \tag{60}$$

and

$$\min R = n + \frac{1}{\varphi_0}\left(1 - \frac{1}{2\varphi_0 n}\right) + \underline{O}(n^{-2}), \tag{61}$$

$$Q(p_1) = \frac{\lambda_1}{\varphi_0\lambda\gamma_1} e^{-a_2\varphi_1'}\left(1 + \frac{1}{n}(b_4 - b_{41} - \frac{1}{2\varphi_0}) + \underline{O}(n^{-2})\right)\frac{1}{N-n} \tag{62}$$

where $a_2 = -(2-p_0)/3$, $b_{4j} = b_4(p_0,p_j)$, $\lambda = \lambda_1 e^{-a_2\varphi_1'} + \lambda_2 e^{-a_2\varphi_2'}$, and

$b_4 = (b_{41}\lambda_1 e^{-a_2\varphi_1'} + b_{42}\lambda_2 e^{-a_2\varphi_2'})/\lambda$.

<u>Proof.</u> The first part of the theorem follows from (56) for $\beta = 1/2$, i.e. $u = 0$. Let $c/n = p_0 + a_2/n + a_4/n^2$ where a_2 and a_4 are defined by (59). Then from theorem 1 we have

$$R = n + (N-n)n^{-1/2}\lambda e^{-n\varphi_0 - b_4/n}(1 + \underline{O}(n^{-2}))$$

with the definitions of λ and b_4 as given above. The remaining part of the proof follows from theorem 4.

Since a_4 is small we have practically $c \cong np_0 - 2/3$ and consequently $P(p_2) \cong Q(p_1)$ according to theorem 2.

The IQL plans are the only plans with a fixed risk which lead to a relationship between c and n of the same type as the one for the Bayesian plans. The coefficients a_2 and a_4 are, however, different. As a result we find in the IQL system $P(p_2)/Q(p_1) \cong 1$ whereas this ratio in the Bayesian system equals $\gamma_1\varphi_1'/\gamma_2(-\varphi_2')$.

Because of the relation $Q(p_1) \cong P(p_2)$ we have

$$R \cong n + (N-n)(\gamma_1 + \gamma_2)Q(p_1) \tag{63}$$

so that we may use the IQL plans defined by $(p_0, p_1, \gamma_1 + \gamma_2)$ as an approximation to the IQL plans defined by $(p_1, p_2, \gamma_1, \gamma_2)$ which makes a tabulation much easier, see [12]. Plans for $\gamma_1 + \gamma_2 \neq 1$ may be found with good approximation from a table for $\gamma_1 + \gamma_2 = 1$ by using $N^* = N(\gamma_1 + \gamma_2)$ as argument.

The IQL plans considered here are generalizations of the plans discussed by Weibull [17] and tabulated by Markback [15] in the same manner as the LTPD plans are generalizations of the Dodge-Romig plans.

<u>10. Minimum average costs for decreasing consumer's or producer's risk.</u>

The restriction imposed is of the form $P(p_2) = \beta(N)$, say, where $\beta(N)$ is a given decreasing function of N tending to zero.

We shall only treat the case $P(p_2) = \beta_0$ for $N \leq N_0$ and $P(p_2) = \beta_0 N_0/N = \beta/N$ for $N > N_0$. For a specified decreasing producer's risk we get analogous results.

For $N \leq N_0$ the theory has already been given in section 8. For $N > N_0$ the procedure for obtaining the exact solution is analogous to the one derived in section 8 with the (numerical) complication that n, as defined by the restriction, becomes a function of both c and N. Therefore the auxiliary function N_c has to be determined by iteration for each c.

The asymptotic solution is given in

Theorem 8. The relation between acceptance number and sample size is $c = np_0 + a_2 + a_4/n + \underline{O}(n^{-2})$ where

$$a_2 = \frac{1}{\delta'} \ln \frac{\lambda_1 \gamma_2 \varphi_0 \delta' \beta}{\lambda_2(-\varphi_2')} \quad \text{and} \quad a_4 = \frac{1}{\delta'}(\beta_2 - \beta_1 + \frac{1}{2\varphi_0}). \tag{64}$$

p_0, φ_0, β_1, and β_2 being defined in (44), (50), and (51), respectively. The relation between sample size and lot size is given by

$$\ln N = \varphi_0 n + \frac{1}{2} \ln n + a_2\varphi_2' + \ln \frac{\beta\gamma_2}{\lambda_2} + (a_4\varphi_2' + \beta_2)\frac{1}{n} + \underline{O}(n^{-2}) \tag{65}$$

and

$$\min R = n + \beta\gamma_2 + \frac{(-\varphi_2')}{\varphi_0 \delta'}(1 - \frac{1}{2\varphi_0 n}) + \underline{O}(n^{-2}). \tag{66}$$

Proof. Writing $c/n = p_0 + \varepsilon$ it follows as in section 6 that $p_1 < p_0 < p_2$. From

$$B(c,n,p_2) = \frac{\lambda_2}{\gamma_2} n^{-1/2} e^{-n\varphi(p_0,p_2) - n\varepsilon\varphi_2'} = \frac{\beta}{N},$$

disregarding terms of "higher order", we find

$$\varphi(p_0,p_2)n + \frac{1}{2}\ln n + n\varepsilon\varphi_2' + \ln(\beta\gamma_2/\lambda_2) = \ln N. \tag{67}$$

For given values of p_0 and ε this gives a relation between n and N. It remains to determine p_0 and ε so that R is minimized.

Writing

$$R = n + (N-n)\gamma_2 P(p_2)((\gamma_1 Q(p_1))/(\gamma_2 P(p_2)) + 1)$$

we find

$$R = n + (1 - \frac{n}{N})\beta\gamma_2 \left(\frac{\lambda_1}{\lambda_2} e^{-n\delta_0 - n\varepsilon\delta'} + 1\right).$$

From (57) we have $n \sim (\ln N)/\varphi(p_0,p_2)$ so that p_0 enters into R through n and $\exp(-n\delta_0)$. To determine the optimum value of p_0 consider first $\delta_0 = 0$ which gives p_0 as determined by (44). As shown below we then determine ε so that $R = n + \underline{O}(1)$. For $\delta_0 < 0$ we get $e^{-n\delta_0} = \underline{O}(N^{\mu})$, $\mu > 0$, which gives a larger R than for $\delta_0 = 0$. For $\delta_0 > 0$ we find $R \sim n + \underline{O}(1)$ but since $\varphi(p_0,p_2) < \varphi_0$ the corresponding R will be larger than for $\delta_0 = 0$. Consequently we have $\delta_0 = 0$.

To determine ε we solve the equation $dR/d\varepsilon = 0$ taking into account that n according to (57) depends on ε. Using $\partial n/\partial\varepsilon \sim n\varphi_2'/\varphi_0$ and solving for $a_2 = n\varepsilon$ we get the result given in (64).

To get one more term we introduce $c/n = p_0 + (a_2 + \varepsilon)/n$ and repeat the procedure to determine the new ε. From

$$\varphi_0 n + \frac{1}{2}\ln n + a_2\varphi_2' + \ln(\beta\gamma_2/\lambda_2) + (\beta_2 + n\varepsilon\varphi_2')n^{-1} = \ln N$$

and

$$R = n + (1 - \frac{n}{N})\beta\gamma_2\left(\frac{\lambda_1}{\lambda_2} e^{a_2\delta' - (\beta_1 - \beta_2 + n\varepsilon\delta')/n} + 1\right)$$

we get $\partial n/\partial\varepsilon = (-\varphi_2'/\varphi_0)(1 - 1/2\varphi_0 n)$ and solving the equation $dR/d\varepsilon = 0$ with respect to $n\varepsilon$ we find a_4 as given in (64).

The remaining part of the theorem follows by inserting the values of a_2 and a_4 in R.

This system of plans has thus the same asymptotic properties as the Bayesian plans. The essential difference between the two systems arises from the different constant terms in the asymptotic relations between c and n and between n and N because these constants depend on the (arbitrary) parameter β.

We may minimize min R with respect to β. Taking into account that n, a_2, a_4, and β_2 are functions of β we find the optimum value is $\beta = \varphi_1'/(\varphi_0\gamma_2\delta')$ as might be expected from the Bayesian solution and furthermore $\min R = n + 1/\varphi_0 + \underline{O}(n^{-1})$.

11. Minimum average costs for a fixed ratio of the consumer's to the producer's risk.

The restriction is given as $P(p_2) = \rho Q(p_1)$ which naturally may be modified so that for $N \leq N_0$ it is further required that $P(p_2) = \beta$, say, and that $P(p_2) \leq \beta$ for $N \geq N_0$.

The restriction defines n as a function of c and the exact solution may therefore be derived by the same method as used in section 8. Since $P(p_1) > P(p_2)$ it is necessary for a solution to exist that $P(p_2) < \rho/(1+\rho)$.

The asymptotic solution is given by

Theorem 9. The relation between acceptance number and sample size is $c = np_0 + a_2 + a_4/n + \underline{O}(n^{-2})$ where

$$a_2 = \frac{1}{\delta'} \ln \frac{\lambda_1 \gamma_2 \rho}{\lambda_2 \gamma_1}, \tag{68}$$

p_0 and a_4 being defined in (44) and (30), respectively.

The relation between sample size and lot size is given by

$$\ln(N-n) = \varphi_0 n + \frac{1}{2}\ln n + a_2\varphi_1' - \ln(\lambda\varphi_0) + (a_4\varphi_1' + \beta_1 - \frac{1}{2\varphi_0})\frac{1}{n} + \underline{O}(n^{-2}) \tag{69}$$

and

$$\min R = n + \frac{1}{\varphi_0}(1 - \frac{1}{2\varphi_0 n}) + \underline{O}(n^{-2}) \tag{70}$$

where $\lambda = \lambda_1(\gamma_1 + \rho\gamma_2)/\gamma_1$ and β_1 is defined in (51).

Proof. The proof is similar to the one in section 6 making use of theorems 3 and 4.

Minimizing min R with respect to ρ gives $\rho = (\gamma_1\varphi_1')/(-\gamma_2\varphi_2')$ which leads back to the Bayesian solution.

For $\rho = 1$ we find for small p_1 and p_2 that $a_2 \approx -2/3$ which means that we get approximately the IQL plans discussed in section 9.

12. Sampling plans defined by two risks.

Suppose now that the weights of the prior distribution and the costs are unknown. We then ask the question: Is it possible from knowledge of (p_1, p_2) alone to define systems of sampling plans having similar properties as the Bayesian and the restricted Bayesian plans?

By means of the asymptotic properties derived in the previous sections we may construct three systems with the required properties, each of them being defined by specifying two risks:

A. Decreasing producer's and consumer's risk, i.e. $Q(p_1) = \alpha/N$ and $P(p_2) = \beta/N$, which corresponds to the Bayesian plans in section 6 and the restricted Bayesian plans in section 10 and 11.

B. LTPD plans with decreasing producer's risk (or AQL plans with decreasing consumer's risk), i.e. $Q(p_1) = \alpha/N$ and $P(p_2) = \beta$ (or $Q(p_1) = \alpha$ and $P(p_2) = \beta/N$), which corresponds to the plans discussed in section 8.

C. <u>IQL plans with decreasing producer's (or consumer's) risk,</u> i.e. $P(p_0) = 1/2$ and $Q(p_1) = \alpha/N$ (or $P(p_2) = \beta/N$), which corresponds to the IQL plans discussed in section 9.

It may be necessary to modify conditions of the form $Q(p_1) = \alpha/N$, say, to $Q(p_1) = \alpha_0$ for $N \leq N_0$ and $Q(p_1) = \alpha_0 N_0/N = \alpha/N$ for $N > N_0$.

It will be noted that the requirement min R has been replaced by conditions as $Q(p_1) = \alpha/N$ and/or $P(p_2) = \beta/N$.

We shall give a short discussion of these three systems. Each of the first two contains two arbitrary parameters, α and β, whereas the last contains only one.

A. From the two conditions we find that $B(c,n,p_2)/(1 - B(c,n,p_1)) = \beta/\alpha$, which defines a relationship between n and c, $n = n_c$ say. Defining the auxiliary function $N_c = \beta/B(c,n_c,p_2)$ it follows that (c,n_c) satisfies the conditions for $N_{c-1} < N < N_c$ (if we interpret the equalities as $\leq$). It is therefore rather easy to tabulate these plans when n_c has been found. Since $P(p_1) > P(p_2)$ it is necessary for a solution to exist that $N > \alpha + \beta$.

Asymptotically we find that $c/n = p_0 + a_2/n + a_4/n^2 + \underline{O}(n^{-3})$ where p_0, a_2, and a_4 are determined by means of theorem 3, i.e.

$$a_2 = \frac{1}{\delta'} \ln \frac{\beta\lambda_1\gamma_2}{\alpha\lambda_2\gamma_1} . \tag{71}$$

The relation between lot size and sample size is given by (55).

The corresponding costs are

$$R = n + (\gamma_1\alpha + \gamma_2\beta) + \underline{O}(n^{-2}) \tag{72}$$

which is minimized with respect to α and β for $\alpha = (-\varphi_2')/\gamma_1\varphi_0\delta'$ and $\beta = \varphi_1'/\gamma_2\varphi_0\delta'$, compare theorem 5.

For any α and β we may thus construct a system of sampling plans with the same asymptotic properties as the Bayesian plans. Tabulation of the 'optimum' values of α and β for typical values of $(p_1,p_2,\gamma_1,\gamma_2)$ may give some guidance for the choice of α and β.

B. The condition $P(p_2) = \beta$ defines $n = n_c$ which together with the other condition gives

$N_c = \alpha/(1 - B(c,n_c,p_1))$. It is necessary that $N > \alpha/(1 - \beta)$.

Asymptotically c/n is given by (56), and by means of theorem 1 we get

$$\ln N = \varphi(p_2,p_1)n + b_1\sqrt{n} + \frac{1}{2}\ln n + b_2 + \ln(\alpha(p_2-p_1)\sqrt{2\pi p_2 q_2/p_1 q_2}) + \ldots \tag{73}$$

where $b_i = b_i(p_2,p_1)$.

The corresponding costs become

$$R = (1-\gamma_2\beta)n + \gamma_1\alpha + \gamma_2\beta N + \underline{O}(n^{-2}) \tag{74}$$

which is minimized with respect to α for $\alpha = (1-\gamma_2\beta)/\gamma_1\varphi(p_2,p_1)$, compare theorem 6.

The system may be characterized as an LTPD system with the consumer's risk inversely proportional to lot size. It is much easier to tabulate than the corresponding restricted Bayesian system.

C. The exact solution is analogous to the one under B.

Asymptotically we have $c/n = p_0 + a_2/n + a_4/n^2 + \underline{O}(n^{-3})$, where the coefficients a_2 and a_4 are given by (59). From theorem 1 we find

$$\ln N = \varphi_0 n + \frac{1}{2}\ln n + a_2\varphi_1' + \ln(\alpha\gamma_1/\lambda_1) + b_4(p_0,p_1)n^{-1} + \underline{O}(n^{-2}). \tag{75}$$

The costs become

$$R = n + \gamma_1\alpha + \frac{\gamma_1\alpha\lambda_2}{\lambda_1} e^{a_2\delta'} (1 + (a_4\delta' + \beta_2 - \beta_1)n^{-1}) + \underline{O}(n^{-2}) \tag{76}$$

which is minimized with respect to α for $\alpha = \lambda_1 e^{-a_2\varphi_1'}/\varphi_0\lambda\gamma_1$, compare theorem 7.

The system is an IQL system with the consumer's risk inversely proportional to lot size. According to theorem 2 the producer's and the consumer's risk are nearly equal.

The exact solution is very easy to tabulate because n_c may be found from the asymptotic solution with sufficient accuracy for $c \geq 1$.

D. Fixed risks. For the sake of completeness we mention the system with fixed risks, i.e. $Q(p_1) = \alpha$ and $P(p_2) = \beta$, which leads to a sampling plan independent of N and gives

$$R = (1-\gamma_1\alpha-\gamma_2\beta)n + (\gamma_1\alpha+\gamma_2\beta)N. \tag{77}$$

For systems A and C the costs consist of sampling inspection costs, $n = \underline{O}(\ln N)$, plus an asymptotically constant average decision loss.

For system B the sampling inspection costs are $\underline{O}(\ln N)$ and the decision loss consists of a constant part (from the decreasing risk) and a part proportional to N (from the constant risk).

For system D sampling inspection costs are constant and the decision loss is proportional to N.

E. Percentage inspection. The plans are defined by $n = \mu N$, μ being a (arbitrary) constant, and $c = p_0 n$ (or $c = p_0 n + a_2 + \ldots$).

From theorem 1 we get

$$R = \mu N + \underline{O}(Ne^{-N}) \tag{78}$$

so that we have the interesting result that for the given model plans with fixed risks and plans with percentage inspection asymptotically have the same costs for $\mu = \gamma_1 \alpha + \gamma_2 \beta$.

In the first case the costs are essentially decision losses (because the sample is too small) and in the second case the costs are essentially inspection costs (because the sample is too large).

For $\gamma_1 = 1$, $\gamma_2 = 0.5$, $\alpha = 0.05$, and $\beta = 0.10$ we get $\mu = 0.10$ which is of the order of magnitude as previously found in practice.

13. Efficiency and robustness.

In a previous paper |10| it has been proposed to define the efficiency of a sampling plan as

$$e(N,n,c) = R_0(N)/R(N,n,c) \tag{79}$$

where $R_0(N)$ denotes the costs of the optimum plan and $R(N,n,c)$ denotes the costs of the plan in question.

For a lot containing X defectives acceptance without inspection is cheaper than rejection without inspection if $X \leq [Np_r]$. Classifying all lots in this way the average costs become

$$K_m(N) = \sum_{X=0}^{[Np_r]} (NA_1 + XA_2) f_N(X) + \sum_{X=[Np_r]+1}^{N} (NR_1 + XR_2) f_N(X).$$

By means of theorem 1

$$K_m(N)/N = k_m + \underline{O}(N^{-3/2}\, e^{-N})$$

and $K_m(N)/N \longrightarrow k_m$ from below.

It would be more correct to define efficiency as the ratio of costs in excess of $K_m(N)$ instead of Nk_m. The difference in the two definitions is, however, of importance only for very small N. The definition (79) tends to underestimate the efficiency.

Theorem 10. Let a system of sampling plans be defined by the two relations $c = np_0 + a_2 + o(1)$ and

$$\ln(N-n) = \varphi_0 n + \frac{1}{2}\ln n + \kappa + o(1) \qquad (30)$$

where $p_0 = \left(\ln \frac{q_1}{q_2}\right) \Big/ \left(\ln \frac{p_2 q_1}{p_1 q_2}\right)$ and $\varphi_0 = p_0 \ln \frac{p_0}{p_i} + q_0 \ln \frac{q_0}{q_i}$, $i = 1$ or 2. Then

$$e(N,n,c) = 1 - \frac{1}{\varphi_0 n}(\varphi_0\lambda + \kappa_0 - \kappa - 1) + o(n^{-1}) \qquad (31)$$

where κ_0 denotes the constant term of (47) and $\lambda = \lambda_1 e^{\kappa - a_2\varphi_1} + \lambda_2 e^{\kappa - a_2\varphi_2}$.

Proof. From theorem 5 we have that $R_0(N) = n_0 + 1/\varphi_0$ where n_0 is defined by (80) for $\kappa = \kappa_0$. By means of theorem 1 and (80) we find that $R(N,n,c) = n + \lambda$ so that

$$e = \left(1 + \frac{n_0 - n}{n} + \frac{1}{\varphi_0 n}\right) \Big/ \left(1 + \frac{\lambda}{n}\right) + o(n^{-1}).$$

Comparison of (80) and the corresponding equation defining n_0 leads to $n_0/n = 1 + (\kappa - \kappa_0)/\varphi_0 n + o(n^{-1})$ which completes the proof.

The theorem shows that the restricted Bayesian plans, apart from the one having a fixed risk, and the corresponding plans based on two risks all have asymptotic efficiency 1. The reasons for this important result are that these plans only differ with respect to the constant terms a_2 and κ.

It should also be noted that p_0 and φ_0 depend on (p_1, p_2) only, whereas a_2 and κ depend on the other parameters also. This means that wrong values of (w_1, w_2) and the cost parameters have a secondary influence on the efficiency which tends to 1 if only (p_1, p_2) are correct.

Since plans having the consumer's and/or the producer's risk fixed, lead to costs of order N, the efficiency of such plans tends to zero as $(\ln N)/N$.

Suppose now that plans have been constructed from wrong values of the parameters, $(p_1^*, p_2^*, \gamma_1^*, \gamma_2^*)$ say, where $p_1^* \neq p_1$ and $p_2^* \neq p_2$. We then get

Theorem 11. Let a system of sampling plans be defined by the two relations $c = np_0^* + a_2^* + o(1)$ and

$$\ln(N-n) = \varphi_0^* n + \frac{1}{2}\ln n + \kappa^* + o(1). \qquad (32)$$

Then

$$e(N,n,c) = \frac{\varphi_0^*}{\varphi_0}\left(1 - \frac{1}{\varphi_0^* n}\left(\kappa_0 - \kappa^* - 1 + \frac{1}{2}\ln \frac{\varphi_0^*}{\varphi_0}\right)\right) + o(n^{-1}) \qquad (33)$$

if and only if $p_1 < p_1^* < p_2^* < p_2$. Otherwise $e \to 0$ as $N^{-\varepsilon}\ln N$ for $0 < \varepsilon \leq 1$.

<u>Proof.</u> If $p_1 < p_0^* < p_2$ if follows from theorem 1 that

$$R = n + (N-n)n^{-1/2} \Sigma\lambda_i^* e^{-n\varphi(p_0^*, p_i) - a_2^* \varphi_i'}$$

$$= n + \Sigma\lambda_i^* e^{-n\delta_i + \kappa^* - a_2^* \varphi_i'}$$

where $\delta_i = \varphi(p_0^*, p_i) - \varphi(p_0^*, p_i^*)$ since $\varphi_0^* = \varphi(p_0^*, p_1^*) = \varphi(p_0^*, p_2^*)$. If $p_0^* \notin (p_1, p_2)$ then R contains a term proportional to (N-n) beside terms as above.

Since

$$\delta_1 > 0 \text{ for } \begin{cases} p_1 < p_1^* \\ p_1 > p_2^* \end{cases} \quad \text{and} \quad \delta_2 > 0 \text{ for } \begin{cases} p_2 > p_2^* \\ p_2 < p_1^* \end{cases}$$

we get $\delta_1 > 0$ and $\delta_2 > 0$ for (1) $p_1 < p_1^* < p_2^* < p_2$, (2) $p_1 < p_2 < p_1^* < p_2^*$, (3) $p_1^* < p_2^* < p_1 < p_2$; $\delta_1 > 0$ and $\delta_2 < 0$ for (4) $p_1 < p_1^* < p_2 < p_2^*$; $\delta_1 < 0$ and $\delta_2 > 0$ for (5) $p_1^* < p_1 < p_2^* < p_2$; $\delta_1 < 0$ and $\delta_2 < 0$ for (6) $p_1^* < p_1 < p_2 < p_2^*$.

In cases (2) and (3) we have $p_0^* \notin (p_1, p_2)$ wherefore $R = \underline{O}(N)$. In cases (4)-(6) we may have $p_0^* \in (p_1, p_2)$, but since at least one of the δ's is negative, and $\delta < 0$ gives

$$\exp(-n\delta) = \exp(-\frac{\delta}{\varphi_0^*} \ln N + \ldots) = \underline{O}(N^{\varepsilon}), \quad 0 < \varepsilon < 1,$$

we get $R = \underline{O}(N^{\varepsilon})$, $0 < \varepsilon \leq 1$, including the cases with $p_0^* \notin (p_1, p_2)$. It follows that $e \to 0$ as $(\ln N)/N^{\varepsilon}$.

In case (1) we have $R = n + \underline{O}(e^{-n})$ so that $e \sim (n_0 + 1/\varphi_0)/n$. From (82) and the corresponding expression for n_0 we find

$$\frac{n_0}{n} = \frac{\varphi_0^*}{\varphi_0} + \left(\kappa^* - \kappa_0 - \frac{1}{2} \ln \frac{\varphi_0^*}{\varphi_0}\right) \frac{1}{\varphi_0 n} + \underline{O}(n^{-2})$$

which leads to (83).

It is also clear that the cases with at least one risk fixed give $e = \underline{O}(N^{-1} \ln N)$.

The general conclusion is that the Bayesian plans (and also the restricted Bayesian plans with decreasing risks) are rather robust if only $p_1 < p_1^* < p_2^* < p_2$. A discussion with numerical examples has been given in [11].

If the prior distribution of p is continuous we get asymptotically that n is proportional to $\sqrt{N}$. This case has been discussed in [10] where also a comparison with IQL plans has been given.

14. An example.

As an example consider a case with $p_1 = 0.01$, $w_1 = 0.85$, and $p_2 = 0.05$, $w_2 = 0.15$. Let the costs of sampling inspection be 0.40 (economic units) per item in the sample, i.e. $S_1 = 0.40$ and $S_2 = 0$, the costs of rejection 0.30 per item in the remainder, i.e. $R_1 = 0.30$ and $R_2 = 0$, and the costs of acceptance per defective item 10.00, i.e. $A_1 = 0$ and $A_2 = 10.00$. It follows that $p_r = 0.03$, $\gamma_1 = 0.6296$, and $\gamma_2 = 0.1111$.

In the table we have compared plans from 8 systems defined as follows:

(1). Bayes. Plans minimizing $R(N,n,c)$.

(2). IQL. Min R for $P(p_0) = 1/2$. We get $p_0 = 0.0250$. The plans have been found by minimizing $R = n + (N-n)(\gamma_1 + \rho\gamma_2)$ where

$$\rho = \frac{\lambda_2\gamma_1}{\lambda_1\gamma_2}\left(\frac{p_2q_1}{p_1q_2}\right)^{-2/3} = 0.9982, \text{ and } \gamma_1+\rho\gamma_2 = 0.74.$$

(3). LTPD. Min R for $P(p_2) = 0.10$. $\gamma = 0.64$.

(4). AQL. Min R for $Q(p_1) = 0.05$. $\gamma = 0.11$.

(5). Fixed risk. $Q(p_1) = 0.05$ and $P(p_2) = 0.10$.

(6). Percentage inspection. $\mu = 0.05\gamma_1 + 0.10\gamma_2 = 0.04259$.

(7). Dodge. The AQL system with 5% consumer's risk proposed by Dodge in [3] with AQL = p_1.

(8). Mil-Std. Military Standard 105D [16] with AQL = p_1.

The Dodge-Romig system has not been included in the comparisons because it gives nearly the same result as the LTPD system with $\gamma = 0.64$.

For each of 7 lot sizes the Bayesian plan and the corresponding costs have been found and for the other systems the efficiency has been computed from (79).

Since γ_1 is considerably larger than γ_2 it will be expected that plans with small values of $Q(p_1)$ as compared to $P(p_2)$ are to be preferred. Systems with a fixed producer's risk, such as (4), (5), and (7), may therefore be expected to give low efficiencies for large lots, as seen in the table.

It is interesting to compare the two risks for the various systems.

The table reveals what price must be paid to obtain a specified degree of protection. It may in some circumstances seem reasonable to pay such a price for small lots, but certainly not for large lots where the protection obtained in terms of the two risks is very good both for the Bayesian and the IQL system.

Comparisons of sampling plans

	Bayes			IQL			LTPD			AQL			Fixed Risk			Percentage Insp.			Dodge			Mil - Std		
N	n	c	R	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e	n	c	100e
100	accept		11	27	0	28	45	0	19	5	0	69	-	-	-	4	0	74	40	1	25	13	0	45
300	accept		33	27	0	44	77	1	32	5	0	84	133	3	24	13	0	65	40	1	54	50	1	46
1000	accept		111	57	1	66	105	2	67	35	1	97	133	3	67	43	1	90	100	2	71	80	2	85
3000	165	3	217	187	4	80	158	4	95	82	2	89	133	3	88	128	3	91	150	3	75	125	3	93
10000	220	6	344	307	7	85	209	6	98	137	3	64	133	3	56	425	11	77	250	5	56	200	5	94
30000	340	9	465	427	10	89	234	7	77	262	5	38	133	3	36	1278	32	36	400	7	35	315	7	74
100000	495	13	501	547	13	91	282	9	42	329	5	17	133	3	15	4259	106	14	650	10	13	510	10	45

N	Bayes $100Q_1$	Bayes $100P_2$	IQL $100Q_1$	IQL $100P_2$	LTPD $100Q_1$	LTPD $100P_2$	AQL $100Q_1$	AQL $100P_2$	Fixed Risk $100Q_1$	Fixed Risk $100P_2$	Percentage Insp. $100Q_1$	Percentage Insp. $100P_2$	Dodge $100Q_1$	Dodge $100P_2$	Mil - Std $100Q_1$	Mil - Std $100P_2$
100	0.0	100.0	23.8	25.0	36.4	9.9	4.9	77.4	-	-	3.9	81.5	6.1	39.9	12.2	51.3
300	0.0	100.0	23.8	25.0	18.0	9.7	4.9	77.4	4.5	9.6	12.2	51.3	6.1	39.9	8.9	27.9
1000	0.0	100.0	14.5	14.5	8.9	9.9	4.8	47.2	4.5	9.6	6.9	36.0	7.9	11.8	4.7	23.1
3000	2.2	22.4	4.1	4.1	2.2	10.0	5.0	21.6	4.5	9.6	4.0	11.3	5.5	5.5	3.7	12.4
10000	0.71	7.3	1.3	1.3	0.54	9.8	5.0	8.4	4.5	9.6	0.15	0.96	4.1	1.3	1.6	6.2
30000	0.26	2.3	0.44	0.44	0.27	9.8	5.0	0.88	4.5	9.6	$0.0^{3}4$	$0.0^{3}5$	5.0	0.062	1.5	1.0
100000	0.059	0.63	0.15	0.15	0.064	9.9	4.9	0.25	4.5	9.6	$0.0^{4}5$	$0.0^{3}3$	6.5	$0.0^{3}25$	1.3	0.046

15. Miscellaneous remarks.

A. Generalizations of the model. As indicated in section 2 the model may be generalized by introducing a polynomial cost function as in (12) and/ or a more general prior distribution. The asymptotic results hold for any prior distribution having probability zero for values of p in the open interval (p_1, p_2), where $p_1 < p_r < p_2$, and assigning finite probabilities, w_1 and w_2, say, $w_1 + w_2 \leq 1$, to the end-points of this interval, i.e. the prior distribution may be arbitrary outside the closed interval (p_1, p_2), see [7], [8], and [11]. Another possible generalization will be to make the prior distribution a function of N, see (13), for example to use a double binomial with p_2/p_1 depending on N.

B. The AOQL system. If $p_1 < p_L < p_2$, p_L denoting the AOQL, it may be shown as in [13] that $R = \mu + \gamma_1/\varphi_1 + \underline{O}(N^{1-\varphi_2/\varphi_1})$, where $\varphi_i = \varphi(p_L, p_i)$ for $i = 1,2$. For $p_L \lesseqgtr p_0$ we have $\varphi_1 \lesseqgtr \varphi_2$ which means that the AOQL system gives a satisfactory result within the given framework only if $p_L \leq p_0$. A generalization of the AOQL system has been discussed in [12].

C. The relation to hypothesis testing. The producer's and consumer's risks correspond to the probabilities of errors of the first and second kind, respectively, for testing the hypothesis $p = p_1$ against the alternative $p = p_2$. Lehmann [14] has suggested to use the relationship $\beta =$ [illegible]α as a rule of thumb for obtaining a reasonable balance between the two error probabilities ($\alpha = Q(p_1)$ and $\beta = P(p_2)$) instead of just using a standard significance level for α. It is interesting to notice that the Bayesian solution asymptotically has this property , see (52), so that Lehmann's rule is supported not only from the minimax point of view, as mentioned by himself, but also from the Bayesian.

Acknowledgement.

My thanks are due to Professor D. Dugué, l'Institut de Statistique de l'Universite de Paris, who invited me to give three lectures on the theory of sampling inspection in April 1965. The present paper is a revised version of the lectures.

My thanks are due to Mr. N. Keiding and Mr. P. Thyregod for checking the asymptotic expansions. Mr. Thyregod also carried out the computations for the example.

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